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Some Applications of Transfer Theorems

Tomoyuki Yoshida
(Hokkaido Univ.)

Section 1. Preliminary results. This section is the abstract of my paper "Character-theoretic transfers".

Definition 1. For groups $H \leq G$ and $\lambda \in \hat{H}$, we set

$$T^G(\lambda) = T_H^G(\lambda) = \det(\lambda^G + 1_H^G).$$

The mapping $T_H^G : \hat{H} \longrightarrow \hat{G}$ is called a character-theoretic transfer.

Lemma 1. Let $H \leq G$. Then the following hold :

(1) $T_H^G : \hat{H} \longrightarrow \hat{G}$ is a homomorphism.

(2) If $H \leq K \leq G$, then $T_K^G \circ T_H^K = T_H^G$.

(3) $T_H^G(\nu_{|H}) = \nu^{|G:H|}$ for any $\nu \in \hat{G}$.

Lemma 2. (Mackey decomposition) Let $H, K \leq G$ and $\lambda \in \hat{H}$. Then

$$T^G(\lambda)_{|K} = \prod_g T^K(\lambda^{g^{-1}}{}_{|K \cap H^g}),$$

where g runs over a complete set of representatives of (H, K)-double cosets of G.

Lemma 3. A character-theoretic transfer is the dual of an ordinary transfer $G/G' \longrightarrow H/H'$.

Lemma 4. Let $H \leq G$, $x \in G$ and $\lambda \in \hat{H}$. Assume $T^G(\lambda)(x) \neq 1$. Then there is g in G such that $\langle x^g \rangle \cap H \not\leq \operatorname{Ker} \lambda$.

Lemma 5. Let H be a subgroup of G of index prime to p. Let $R_H^G : \hat{G}_p \longrightarrow \hat{H}_p$ be the restriction, where $\hat{G}_p$ and $\hat{H}_p$ are Sylow p-subgroups of $\hat{G}$ and $\hat{H}$, respectively. Then the composition $T_H^G \circ R_H^G : \hat{G}_p \longrightarrow \hat{H}_p \longrightarrow \hat{G}_p$ is an isomorphism and $\hat{H}_p = (\operatorname{Im} R_H^G) \times (\operatorname{Ker} T_H^G)$.

Lemma 6. (Tate) Let P be a Sylow p-subgroup of G and let $K \leq P \leq H \leq G$. Assume that $P \cap G^pG' = (P \cap H^pH')K$. Then the following hold :

(1) If $K \leq G'$, then $P \cap G' = (P \cap H')K$.

(2) If $K \leq O^p(G)$ and $K \trianglelefteq P$, then $P \cap O^p(G) = (P \cap O^p(H))K$.

Notation 1. Let X and Y be subsets of a group. Then we set

$$X^Y = \{ x^y \mid x \in X , y \in Y \}, \quad X_Y = \bigcap_{y \in Y} X^y .$$

Definition 2. Let H be a group, x a p-element, $S \leq H$ and $\lambda \in \hat{S}$. Assume that the following hold :

(a) λ is of order p,

(b) $T_S^H(\lambda)(x) \neq 1$, and

(c) $\langle x^p \rangle^H \cap S \subseteq \operatorname{Ker} \lambda$.

Then we say that the triplet (S, λ, x) is a singularity in H. The subgroup S is called a singular subgroup or simply a singularity. The element x is called a singular element. The character λ is called a singular character. We denote the set of singularities (and also singular subgroups) in H by $\operatorname{Sing}(H)$.

Example 1. Let P be a 2-group, $K \leq P$ and t an involution of P. Set $S = N_P(K)$ and suppose t acts on the set P/K as an odd permutation. Then $|S : K| = 2$ and (S, λ, t) is a singularity in P, where λ is a linear character of S with kernel K.

Example 2. Let $\overline{H}$ be a transitive permutation group with the stabilizer $\overline{S}$ of one point. Let C be a cyclic group of order p. Construct the wreath product $H = C \wr \overline{H}$ with base subgroup V ($\cong C^{|\overline{H}:\overline{S}|}$). Then $V\overline{S}$ is a singular subgroup in H with singular element contained in V.

Lemma 7. Let P be a Sylow p-subgroup of G and let $P \leq H \leq G$. Assume $P \cap G' \neq P \cap H'$. Then there are $g \in G - H$, $x \in P$ and a linear character μ of H of order p such that $(S, \lambda, x) \in \mathrm{Sing}(P)$, where $S = P \cap H^g$ and $\lambda = \mu^{g^{-1}}{}_{|S}$.

Lemma 8. Let $(S, \lambda, x) \in \mathrm{Sing}(H)$. Then the following hold :

(1) $(S^a, \lambda^{a^{-1}}, x^b) \in \mathrm{Sing}(H)$ for any $a, b \in H$.

(2) $S - \mathrm{Ker}\,\lambda$ contains a conjugate of x.

(3) If $S \leq R \leq H$, then $(R, T_S^R(\lambda), x) \in \mathrm{Sing}(H)$.

(4) If $S \leq L \leq H$, then $(S, \lambda, y) \in \mathrm{Sing}(L)$ for some $y \in L \cap x^H$.

(5) If $N \trianglelefteq H$ and $N \leq \mathrm{Ker}\,\lambda$, then $(S/N, \lambda, xN) \in \mathrm{Sing}(H/N)$.

(6) If $x \in L \leq H$, then there is a conjugate $R = S^h$ of S such that $(R \cap L, \lambda^{h^{-1}}{}_{|R \cap L}, x) \in \mathrm{Sing}(L)$.

(7) If $P \in \mathrm{Syl}_p(S)$, then $(P, \lambda_{|P}, x) \in \mathrm{Sing}(H)$.

Notation 2. For any p-group P, $\Phi^*(P)$ denotes the intersection of all subgroups of P of index at most p^2.

Definition 3. A p-group which has no quotient groups isomorphic to $Z_p \wr Z_p$ is called a weakly regular p-group.

Lemma 9. Let P be a p-group. Then $cl(P/\Phi^*(P)) \leq p$ and $\Phi^*(P) \geq \Phi(\Phi(P))$. P is weakly regular if and only if $cl(P/\Phi^*(P)) < p$. When $p = 2$, P is weakly regular if and only if all subgroups of P of index 4 are normal.

Lemma 10. A p-group P has no proper singular subgroup if and only if P is weakly regular.

Lemma 11. Let (S, λ, x) be a singularity in a p-group P. Set $K = \mathrm{Ker}\,\lambda$ and $\overline{P} = P/K_P$. Then the following hold :

(1) Assume $x \in V = S_P$. Then $\overline{P} \simeq Z_p \wr (P/V)$, where the wreath product is constructed by the permutation representation of P/V on the set P/S and the base subgroup is $\overline{V}$. In particular, $m(\overline{V}) = |P : S|$ and $\overline{V} = \langle \overline{x}^{\overline{P}} \rangle$.

(2) $N_P(K) = S$.

(3) Let $Q \leq P$ and assume that $[x, y; p-1] \in \Phi^*(Q)$ for all $y \in Q$. Then $\langle Q, x \rangle$ is contained in a conjugate of S.

(4) $N_P(\langle x \rangle)$ is contained in a conjugate of S.

(5) If $N \trianglelefteq P$ and $N \cap S \leq K$, then $N \leq K_P$.

(6) $\overline{S} = \overline{K} \times Z(\overline{P})$ and $|Z(\overline{P})| = p$.

(7) If $S < Q \leq P$, then Q is not weakly regular.

(8) If $S \leq R \trianglelefteq Q \leq P$, then $m(R/\Phi(R)) \geq |Q : R|$.

(9) Let $P = P_0 > P_1 > \cdots > P_n = S$ be a series of subgroups such that $|P_i : P_{i+1}| = p$ for $0 \le i < n$. Then $\mathrm{Ker}\, T^{P_i}(\lambda) \not\le P_{i+1}$ for $0 \le i < n$. Let $a_0 \in P_0 - P_1$ and $a_i \in \mathrm{Ker}\, T^{P_i}(\lambda) - P_{i+1}$ for $0 < i < n$. Define inductively elements x_i, $0 \le i \le n$, by the rule $x_0 = x$ and $x_{i+1} = [x_i, a_i; p-1]$. Then $T^P(\lambda)(x) = \lambda(x_n) \ne 1$.

(10) $\bar{x} \notin Z_{n(p-1)}(\bar{P})$, $cl(\bar{P}) > n(p-1)$.

(11) If $p = 2$, $K_P = 1$ and N is a cyclic normal subgroup of P, then $x^2 \in C_P(N)$ and $\langle x\rangle N/\langle x^2\rangle$ is dihedral or semi-dihedral.

(12) Let N be a subgroup of P normalized by x. Assume that $p = 2$, $\exp(N/N') \le 2^n$, $n \ge 2$, and that all subgroups of N of index at most 2^{n+1} contain N'. Then there is a conjugate T of S such that $|N : N \cap T| < 2^n$.

(13) Assume that P is the central product of P_1 and P_2 and that $x = x_1x_2$, where $x_i \in P_i$. Then for some i, $(S \cap P_i, \lambda|_{S \cap P_i}, x_i) \in \mathrm{Sing}(P_i)$, $(S, \lambda, x_i) \in \mathrm{Sing}(P)$ and $S \ge P_j$, where $j \ne i$.

Theorem 1. If G has a weakly regular Sylow p-subgroup P, then $P \cap G' = P \cap N_G(P)'$.

Theorem 2. Let $P \in \mathrm{Syl}_p(G)$ and let Q be a weakly closed subgroup of P such that $[P, Q; p-1] \le \Phi^*(Q)$, then $P \cap G' = P \cap N_G(Q)'$.

Theorem 3. Let $P \in Syl_p(G)$ and let Q be a strongly closed and weakly regular subgroup of P. Then $P \cap G' = P \cap N_G(Q)'$.

Theorem 4. Let $P \in Syl_p(G)$ and $P \leq H \leq G$. Take elements $x_1, \cdots, x_m$ of P such that $H = H^pH'\langle x_1, \cdots, x_m\rangle$ and x_k is an element of $P - H^pH'\langle x_1, \cdots, x_{k-1}\rangle$ of minimal order for each k. Let G_k, $1 \leq k \leq m$, be the subsets of G which consist of $g \in G - H$ such that $(P \neq)P \cap P^g$ is a singular subgroup in H with singular element x_k. Set $P_k =$ $\langle x_k^{-1}x_k^{Hg^{-1}} \cap P \mid g \in G_k\rangle$, $1 \leq k \leq m$. Then $P \cap G' =$ $P_1 \cdots P_m(P \cap H')[P, N_G(P)]$.

Theorem 5. Let $P \in Syl_p(G)$. Take elements $x_1, \cdots, x_m$ of $P - \Phi(P)$ such that $P = \langle x_1, \cdots, x_m\rangle$ and x_k is of minimal order in $P - \langle x_1, \cdots, x_{k-1}\rangle\Phi(P)$ for each k. Let $\mathfrak{F}$ be the set of pairs (F, N), where $F < P$ and $F \leq N \leq N_G(F)$, satisfying the following conditions :

(a) F is a tame intersection;

(b) $N_G(F)/F$ is p-isolated;

(c) $F \in Syl_p(O_{p'p}(N_G(F)))$;

(d) For any $x \in N_P(F) - F$, $N = \langle x^N, N_P(F)\rangle$;

(e) $N_G(F)$ is p-constrained ;

(f) F contains a conjugate of a singular subgroup in P with singular element x_k for some k ;

(g) $N_P(F)$ has a normal subgroup K such that $\Phi(F) \leq K \leq F$ and $N_P(F)/F \simeq Z_p \wr (N_P(F)/F)$. In particular, $m(F/\Phi(F)) \geq |N_P(F):F|$.

(h) If $p = 2$ and $m(N_P(F)/F) > 1$, then there is $L \trianglelefteq N$

such that $\Phi(F) \leq L < F$, $B = N/C_N(F/L)$ is a Bender group and F/L is the Steinberg module of B .

Then $P \cap G' = [P, N_G(P)] \langle [F, N] \mid (F, N) \in \mathfrak{F} \rangle$.

Section 2. Applications.

Theorem 1. Frobenius kernels are nilpotent.

Theorem 2. Conway's group C_2 is characterized by its Sylow 2-subgroup.

Theorem 3. Radvalis group Rd is characterized by its Sylow 2-subgroups.

Theorem 4. The unipotent subgroups of simple groups of Lie type are weakly regular, except for the following :

$$(*) \quad \begin{array}{l} L_n(2),\ U_n(2),\ Sp(2n, 2),\ D_n(2),\ {}^2D_n(2^2),\ {}^3D_4(2^3), \\ E_n(2),\ {}^2E_6(2),\ F_4(2),\ {}^2F_4(2)'. \end{array}$$

Conjecture 1. Simple groups with weakly regular Sylow 2-subgroup have abelian Sylow 2-subgroups or are isomorphic to simple groups of Lie type of characteristic 2 excepting the above (*).

Conjecture 2. Let $P \in Syl_p(G)$, $p \neq 2$. Let $\mathfrak{R}(P)$ be the set of weakly regular subgroups of P of maximal order and let $\mathfrak{R}^*(P)$ be the set of $R \in \mathfrak{R}(P)$ such that $|R:\Phi(R)|$ is maximal. Set $W(P) = \langle \mathfrak{R}^*(P) \rangle$. Then $P \cap G' = P \cap N_G(W(P))'$.